OST
Ostschweizer
Fachhochschule

1 Relativity at mm/s speed

Systemtechnik BSc
FS 2026

Aufgaben

IAI Blog im Modul The Lorentz Force

The most profound realization is that the **magnetic force is essentially the electric force viewed from a different frame**. The "magnetic" part of the Lorentz force is a relativistic manifestation of the electric field. The failure of the flux rule in unipolar induction highlights the necessity of treating $\vec{E}$ and $\vec{B}$ as components of a single relativistic object, the Faraday tensor. Even at the "low" speeds of a copper wheel, magnetism is fundamentally a relativistic phenomenon.

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1 The Lorentz Force is a Relativistic Effect

The Faraday law of induction is often presented as a fundamental postulate of classical electromagnetism. However, a deeper analysis reveals that the separation between electric and magnetic forces is frame-dependent. This blog demonstrates how the Lorentz force arises naturally from the requirements of Special Relativity (SR), resolving paradoxes where the classical flux rule appears to fail.

The standard expression for induced electromotive force (EMF) is given by the change in magnetic flux Φ:

$$\boxed{U_{\text{ind}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{A}} \tag{1}$$

While this formula is robust for closed loops changing shape or experiencing time-varying fields, it encounters conceptual difficulties in systems where the geometry of the "loop" is ambiguous.

2 Unipolar Induction (The Faraday Disc)

A classic problem that highlights the distinction between the "flux rule" and the Lorentz force is the **Faraday Disc**, also known as a unipolar generator. This device consists of a conducting disc rotating in a uniform, stationary magnetic field $\vec{B}$ parallel to the axis of rotation.
Consider a conductive copper disc (Faraday wheel) rotating in a uniform, constant magnetic field $\vec{B}$ parallel to its axis of rotation.

(a) The magnetic flux Φ through any sector of the disc remains constant because $\vec{B}$ is uniform and the area within the circuit (brushes to center) does not change in the traditional sense: $\frac{d\Phi}{dt} = 0$.

(b) However, a potential difference U_{ind} is measured between the axis and the rim.

The following diagram illustrates the setup. Note that while the disc rotates, the magnetic field and the external circuit (brushes) remain stationary in the laboratory frame.
Recall the integral form of Faraday's Law (the flux rule):

$$U_{\text{ind}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{A} \tag{2}$$

In this setup:

(a) The magnetic field $\vec{B}$ is constant in time ($\partial\vec{B}/\partial t = 0$).

(b) The area S defined by the circuit (axis $\to$ rim $\to$ external wiring) does not change its shape or orientation relative to the $\vec{B}$ field as the disc rotates.

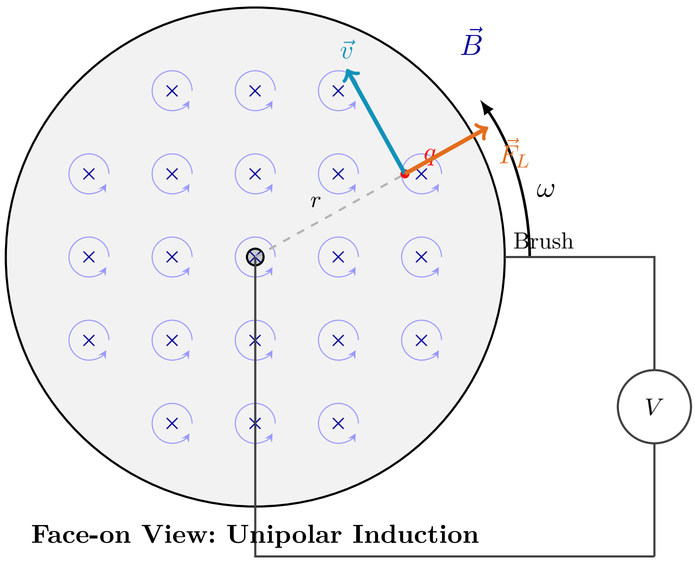

Abbildung 1: Face-on view of the Faraday disc. The blue circular arrows around the crosses represent the magnetic field $\vec{B}$ directed into the page. The Lorentz force $\vec{F}_L = q(\vec{v} \times \vec{B})$ drives charges radially toward the rim.

Consequently, $\frac{d\Phi}{dt} = 0$. Purely based on the flux rule, one would incorrectly predict $U_{\text{ind}} = 0$.

Resolution via the Lorentz Force: The actual induction occurs because the charge carriers (electrons) are moving *with* the disc at velocity $\vec{v}$. At a radial distance r from the center, the velocity is $\vec{v} = \vec{\omega} \times \vec{r}$.

The Lorentz force acting on a charge q is:

$$\vec{F}_L = q(\vec{v} \times \vec{B}) \tag{3}$$

For a disc in the xy-plane and $\vec{B} = B\hat{z}$, the velocity is $\vec{v} = \omega r\hat{\phi}$. The resulting force is:

$$\vec{F}_L = q(\omega r\hat{\phi} \times B\hat{z}) = q\omega rB\hat{r} \tag{4}$$

This radial force acts as a "non-conservative" electric field $\vec{E}_{\text{eff}} = \vec{v} \times \vec{B}$. The induced EMF is the line integral of this field from the axis ($r = 0$) to the rim ($r = R$):

$$U_{\text{ind}} = \int_0^R (\vec{v} \times \vec{B}) \cdot d\vec{r} = \int_0^R \omega rB\,dr = \frac{1}{2}\omega R^2 B \tag{5}$$

The Relativistic Insight

The Faraday Disc paradox arises only if we insist that induction must be caused by a changing magnetic flux. Relativity teaches us that the distinction between "changing flux" and "motion through a field" is frame-dependent. In the lab frame, there is no $\vec{E}$ field, only $\vec{B}$, and the force is magnetic. If we were to sit on a charge element in the disc, we would see a stationary disc but a *transformed* electric field $\vec{E}'$ that accounts for the potential.

3 The Moving Rod Paradox

Consider a conductive rod of length L moving with velocity $\vec{v} = v\hat{x}$ through a constant magnetic field $\vec{B} = B\hat{z}$.

- **Laboratory Frame (S):** The charges in the rod experience a Lorentz force $\vec{F}_m = q(\vec{v} \times \vec{B}) = -qvB\hat{y}$. This causes a charge separation and an induced voltage $U = vBL$.
- **Rod Frame (S'):** In the frame moving with the rod, the rod is stationary ($\vec{v}' = 0$). Since $\vec{B}$ is constant and uniform, $d\Phi/dt = 0$. If we strictly applied classical mechanics, no force should act on the charges, and no voltage should appear.

This discrepancy suggests that the magnetic field in S must manifest as an *electric* field in S'.

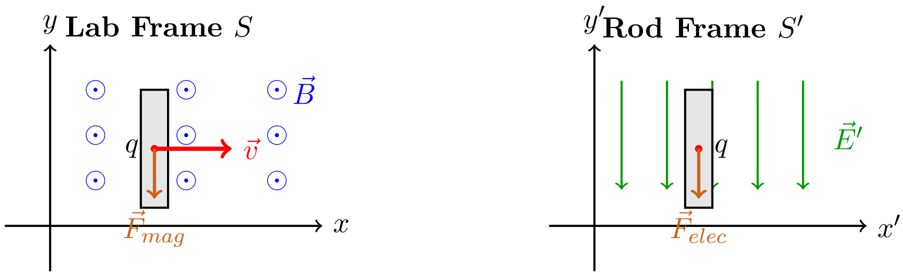

Abbildung 2: In the Lab frame, the force is magnetic. In the Rod frame, the rod is stationary, and the force is purely electric due to the Lorentz transformation of the Faraday tensor.

To resolve the rod paradox, we invoke the Lorentz transformation of the electromagnetic field tensor $F^{\mu\nu}$. For a boost in the x-direction with $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$, the fields transform as:

$$\vec{E}' = \gamma(\vec{E} + \vec{v} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{E}) \tag{6}$$

$$\vec{B}' = \gamma(\vec{B} - \frac{1}{c^2}\vec{v} \times \vec{E}) - \frac{\gamma^2}{\gamma+1}\frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{B}) \tag{7}$$

In the rod's frame (S'), where $\vec{E} = 0$ in the lab, we find:

$$\vec{E}' = \gamma(\vec{v} \times \vec{B}) \tag{8}$$

At low speeds ($v \ll c$, $\gamma \approx 1$), the rod "sees" an electric field $\vec{E}' = \vec{v} \times \vec{B}$. The force in the rod's frame is purely electrostatic: $\vec{F}' = q\vec{E}'$.

4 The Lorentz Boost in Matrix Form

To understand the origin of this force, we must look at how electromagnetic fields transform between inertial frames. We define a boost in the x-direction with velocity v. Let $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. The Lorentz transformation matrix $\Lambda^\mu{}_\nu$ is:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

In relativity, $\vec{E}$ and $\vec{B}$ are not independent vectors but components of the antisymmetric **Faraday Tensor** $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{10}$$

The transformation to a moving frame S' is given by the tensor contraction $F' = \Lambda F \Lambda^T$.

4.1 Deriving the Rod Potential

Consider a conductive rod in the Lab Frame (S) moving at $\vec{v} = v\hat{x}$ in a constant field $\vec{B} = B\hat{z}$. In S, $\vec{E} = 0$.
The only non-zero components of $F^{\mu\nu}$ are $F^{12} = -B$ and $F^{21} = B$.

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

We calculate the new electric field component E'_y using $F'^{02} = -E'_y/c$:

$$F'^{02} = \Lambda^0{}_\alpha F^{\alpha\beta} \Lambda^2{}_\beta \tag{12}$$

$$F'^{02} = \Lambda^0{}_1 F^{12} \Lambda^2{}_2 \quad \text{(since only } \Lambda^2{}_2 = 1 \text{ and } F^{02} = 0\text{)} \tag{13}$$

$$-\frac{E'_y}{c} = (-\beta\gamma) \cdot (-B) \cdot (1) \tag{14}$$

$$E'_y = -\beta\gamma c B = -\gamma v B \tag{15}$$

In the limit of everyday speeds where $v \ll c$:

(a) $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} \approx 1$

(b) Therefore, $E'_y \approx -vB$

Conclusion

In the Lab Frame, we see a magnetic force $\vec{F} = q(\vec{v} \times \vec{B})$ pushing charges to the ends of the rod. In the Rod's Frame, the rod is stationary, but the relativistic transformation creates an **electric field** $\vec{E}' = \vec{v} \times \vec{B}$ that performs the exact same work. **Magnetism is simply the electric force viewed from a moving frame.**

Lösungen